Axisymmetric Potential Flow in a Circular Tube

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Axisymmetric Green's functions and the associated stream functions, which satisfy the boundary condition that the wall of the tube is a stream surface, are presented for various singularities. These include a source on the axis, an axially-oriented doublet, a source ring, a source disk, and a vortex ring. Results are expressed as integrals of the modified Bessel functions. These can be applied to formulate Fredholm integral equations of either the first or second kind for determining the axisymmetric, irrotational flow about bodies of revolution in a tube. In the present work, only those of the first kind are treated, including integral equations for axial source and doublet distributions, and a vortex sheet on the surface of the body. Three different methods for solving each of these three integral equations are examined: the method of piecewise-constant singularity elements of von Kármán, Kaplan's method of expanding the unknown distribution as a series of Legendre polynomials, and solutions by a technique of eliminating peaks in the kernel (representing integrals by means of a quadrature formula) and solving the resulting set of linear equations by means of a suitable iteration formula. Numerical results for three of the methods, applied to a spheroid, are presented. The resulting added masses and a comparison with predictions from slender-body theory are given in Appendixes.

Introduction

FLOWS about bodies in channels of circular section have been studied for the purpose of determining wall corrections for model tests in wind tunnels and ship-model towing tanks. In recent years, several methods of determining the axisymmetric, irrotational flow about bodies of revolution in a tube have been proposed. Levine[1] expressed the basic potentials, for a point source, a ring of sources, and a vortex ring, in terms of an infinite series of Bessel functions of the first kind, with arguments proportional to their zeros, each of which satisfies the boundary condition on the wall of the tube. A second paper, by Mathew,[2] transformed the stream function for a vortex ring, derived from Levine's form of the potential, into infinite integrals of the modified Bessel functions, and used the latter forms to formulate integral equations for velocity distributions in the body surface. A third paper, by Satija,[3] presented a method due to the present author (previously available only in the author's class notes) and applied it to obtain a first-order wall correction for the flow about a spheroid. In a fourth paper,[4] the basic potential for a doublet on the axis, which satisfies the boundary condition at the channel wall, was used to formulate integral equations for an axial doublet distribution and the velocity distribution on the body, and numerical procedures for solving these equations were presented. Goodman[5] applied the same form of the potential for a source as Satija, to obtain flows by slender-body theory.

There are two purposes for the present work. First, the basic potential for a source, used by Satija and Goodman, is in the form of a divergent integral. Although, fortunately, this error does not affect the subsequent equations, it is nevertheless necessary to correct it. Furthermore, Satija has indicated only how to obtain a first-order wall correction; and because the sharp peaks in the kernel of the integral equation were not eliminated, the accuracy of the quadrature formula used to discretize the integral equation would become increasingly poorer the more slender the body. Means for obtaining higher-order solutions and eliminating the peaks of the kernel will be presented.

The second purpose is the broader one of deriving and displaying the various basic potentials which can serve to formulate integral equations for axisymmetric flows in a circular tube, but in terms of integrals of the modified Bessel functions, rather than the form used by Levine[1] which is considered to be computationally less convenient. Mathew[2] succeeded in deriving such a form of the stream function for a vortex ring from that of Levine in a remarkable, but long and tedious transformation; but the associated potential was not presented. In probably the oldest paper on this subject, Dougal[6] derived the Green function for a source in a tube, and also expressed his result as an integral of the modified Bessel functions. This Green function, however, satisfies the boundary condition that the potential, rather than the stream function, is constant on the wall of the tube.

Short and simple derivations of the basic potentials in a tube will be presented for a source and doublet on the axis, and for source and vortex rings. These will be used to formulate various Fredholm integral equations of the first kind, and methods of solving these equations will be discussed. There are several reasons for this limitation to integral equations of the first kind. 1) They are usually avoided since, in general, they do not have exact solutions. Yet if properly treated, they can yield approximate solutions of a high order of accuracy, at a fraction of the computing time usually required by an integral equation of the second kind, for elongated bodies of continuous slope and curvature. 2) Solutions by integral equations of the second kind have been thoroughly treated by Mathew.[2] 3) Without this restriction, this paper would surely become overly long. Numerical results from three of the integral equations will be presented for a spheroid.

Basic Potentials and Stream Functions

A. Source on Axis

We shall employ cylindrical coordinates (x, r, θ) with the x-axis coincident with the axis of the circular tube of radius a. Denote the potential of a source of unit strength at the origin by

$$\Phi = -1/R + \phi(x, r), \quad R = (x^2 + r^2)^{1/2} \tag{1}$$

where $\phi(x,r)$ is the disturbance potential, which satisfies the boundary condition at the wall of the tube

$$\partial\phi/\partial r = (\partial/\partial r)(1/R), \quad r = a \tag{2}$$

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We have[7]

$$\frac{1}{R} = \frac{2}{\pi}\int_0^\infty K_0(kr)\cos kx dk \qquad (3)$$

where $K_0(kr)$ is a modified Bessel function of the first kind. Substituting this form into Eq. (2), then gives

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = -\frac{2}{\pi}\int_0^\infty kK_1(ka)\cos kx dk \qquad (4)$$

since[7]

$$(d/dr)K_0(kr) = -kK_1(kr)$$

This immediately suggests the solution, which is regular and harmonic within the tube and satisfies Eq. (4),

$$\phi(x,r) = \frac{2}{\pi}\int_0^\infty \frac{K_1(ka)}{I_1(ka)}[I_0(ka) - I_0(kr)\cos kx]dk \qquad (5)$$

since[7]

$$(d/dr)I_0(kr) = kI_1(kr)$$

Here I_0 and I_1 denote modified Bessel functions of the second kind. Combining Eqs. (3) and (5), we obtain the expression for the potential

$$\Phi(x,r) = -\frac{2}{\pi}\int_0^\infty \left\{\left[K_0(kr) + \frac{K_1(ka)}{I_1(ka}I_0(kr)\right]\cos kx - \frac{K_1(ka)I_0(ka)}{I_1(ka)}\right\}dk \qquad (6)$$

If the term $I_0(ka)$ in Eq. (5) were omitted, as is the case in the form used by Satija[3] and Goodman,[5] the integral in Eq. (5) would become divergent, since the integral would then vary as k^{-2} near $k = 0$. This additional term plays the role of a constant, however, and does not affect the results given by Satija and Goodman for the stream function and the velocity components.

The associated stream function is given by

$$\Psi(x,r) = -\frac{x}{R} + \frac{2r}{\pi}\int_0^\infty \frac{K_1(ka)}{I_1(ka)}I_1(kr)\sin kx dk \qquad (7)$$

or, from Eq. (3),

$$\Psi(x,r) = -\frac{2r}{\pi}\int_0^\infty \left[K_1(kr) - \frac{K_1(ka)}{I_1(ka)}I_1(kr)\right]\sin kx dk \qquad (8)$$

B. Doublet on Axis

The potential and stream functions for a doublet of unit strength at the origin, oriented in the positive sense of the x-axis, are obtaned immediately as the negative of the derivative with respect of x of the potential and stream functions for a source. We obtain from Eqs. (1) and (5)

$$\Phi = -\frac{x}{R^3} - \frac{2}{\pi}\int_0^\infty \frac{K_1(ka)}{I_1(ka)}I_0(kr)k\sin kx dk \qquad (9)$$

or, from Eq. (6)

$$\Phi = -\frac{2}{\pi}\int_0^\infty \left[K_0(kr) + \frac{K_1(ka)}{I_1(ka)}I_0(kr)\right]k\sin kx dk \qquad (10)$$

and from Eqs. (7) and (8)

$$\Psi(x,r) = \frac{r^2}{R^3} - \frac{2r}{\pi}\int_0^\infty \frac{K_1(ka)}{I_1(ka)}I_1(kr)k\cos kx dk \qquad (11)$$

or

$$\Psi(x,r) = \frac{2r}{\pi}\int_0^\infty \left[K_1(kr) - \frac{K_1(ka)}{I_1(ka}I_1(kr)\right]k\cos kx dk \qquad (12)$$

C. Source Ring

For a source distribution of unit strength on the circle $r = c < a$, situated at $x = 0$, the potential at a point (x,r) may be written in the form

$$\Phi(x,r) = -c\int_0^{2\pi}\frac{d\theta}{R'} + \phi(x,r) \qquad (13)$$

where

$$R' = [x^2 + r^2 + c^2 - 2rc\cos\theta]^{1/2} \qquad (14)$$

is the distance from the point (x,r) to a point on the circle $r = c$. Here the integral in Eq. (13) can be expressed in the form

$$\int_0^{2\pi}\frac{cd\theta}{[x^2 + r^2 + c^2 - 2rc\cos\theta]^{1/2}} = \int_0^{2\pi}\frac{cd\theta}{\left[x^2 + (r+c)^2 - 4rc\cos^2\frac{\theta}{2}\right]^{1/2}} = \frac{4c}{\rho}K(h) \qquad (15)$$

where $K(h)$ is the complete elliptic integral of the first kind, with

$$\rho = [x^2 + (r+c)^2]^{1/2},\quad h = (2/\rho)(rc)^{1/2}$$

Applying the known expansion (see Ref. 8, p. 103)

$$\frac{1}{R'} = \frac{2}{\pi}\int_0^\infty [K_0(kr)I_0(kc) + 2\sum_{n=1}^{\infty}K_n(kr)I_n(kc)\cos n\theta]\cos kx dk \qquad (16)$$

we obtain

$$\int_0^{2\pi}\frac{d\theta}{R'} = 4\int_0^\infty K_0(kr)I_0(kc)\cos kx dk \qquad (17)$$

Also, the disturbance potential $\phi(x,r)$ in Eq. (13) satisfies the boundary condition

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = c\frac{\partial}{\partial r}\int_0^{2\pi}\frac{d\theta}{R'}\bigg|_{r=a} = -4c\int_0^\infty K_1(ka)I_0(kc)k\cos kx dk \qquad (18)$$

This suggests the solution, regular and harmonic within the tube

$$\phi(x,r) = 4c\int_0^\infty \frac{K_1(ka)}{I_1(ka)}I_0(kc)[I_0(ka) - I_0(kr)\cos kx]dk \qquad (19)$$

Hence we have the alternative forms for the potential

$$\Phi(x,r) = -\frac{4c}{\rho}K(h) + 4c\int_0^\infty \frac{K_1(ka)}{I_1(ka)}I_0(kc)[I_0(ka) - I_0(kr)\cos kx]dk \qquad (20)$$

and, from Eq. (17)

$$\Phi(x,r) = -4c\int_0^\infty \left\{\left[K_0(kr) + \frac{K_1(ka)}{I_1(ka)}I_0(kr)\right]\cos kx - \frac{K_1(ka)I_0(ka)}{I_1(ka)}\right\}I_0(kc)dk \qquad (21)$$

The associated stream function is then given by

$$\Psi(x,r) = -4cr\int_0^\infty \left[K_1(kr) - \frac{K_1(ka)}{I_1(ka)}I_1(kr)\right]I_0(kc)\sin kx dk \qquad (22)$$

D. Source Disk

The results for a source distribution of unit strength on the disk bounded by the circle $r = c$, situated at $x = 0$, can be obtained by integrating those for a source ring with respect to c. Since[7]

$$\int_0^c kcI_0(kc)dc = cI_1(kc)$$

we immediately obtain, from Eqs. (21) and (22)

$$\Phi(x, r) = -4c\int_0^\infty \left\{\left[K_0(kr) + \frac{K_1(ka)}{I_1(ka)}I_0(kr)\right]\cos kx - \frac{K_1(ka)I_0(ka)}{I_1(ka)}\right\}\frac{I_1(kc)}{k}dk \quad (23)$$

and

$$\Psi(x, r) = -4cr\int_0^\infty \left[K_1(kr) - \frac{K_1(ka)}{I_1(ka)}I_1(kr)\right]\frac{I_1(kc)}{k}\sin kx dk \quad (24)$$

E. Vortex Ring

The potential of the flow induced by a vortex of unit strength on the circle $r = c$ at $x = 0$ is the same as that of a distribution of doublets, oriented in the x direction, on the disk bounded by $r = c$, of strength $1/4\pi$. Since the results for the doublet disk are given by the negative of the derivative with respect to x of those for the source disk, we obtain immediately, from Eqs. (23) and (24), for the vortex ring

$$\Phi(x, r) = -\frac{c}{\pi}\int_0^\infty \left[K_0(kr) + \frac{K_1(ka)}{I_1(ka)}I_0(kr)\right]I_1(kc)\sin kx dk \quad (25)$$

$$\Psi(x, r) = \frac{cr}{\pi}\int_0^\infty \left[K_1(kr) - \frac{K_1(ka)}{I_1(ka)}I_1(kr)\right]I_1(kc)\cos kx dk \quad (26)$$

Integral Equations

Many different integral equations can be formulated for computing the axisymmetric flow about a body of revolution in a tube. It is usually possible, however, to select one which is well-suited for a particular problem, depending upon the physical quantities of principal interest, the regularity and fineness of the body shape, and considerations of computing economy. Since the basis for a choice will be more substantial and more easily illustrated when a set of integral equations is at hand, let us return to this question after the various integral equations have been derived.

Let $r = r(x)$ be the equation of a body of revolution, its axis extending over $-l_1 \le x \le l_2$ along the x-axis. A uniform stream of unit velocity in the x-direction is incident on the body.

Axial Source Distribution

Assume that a source distribution of strength $m(x)$ extends over the x-axis in the range $-1 \le x \le 1$. For slender bodies, it is usually sufficiently accurate to take the x-coordinates of the upstream and downstream ends of the body as

$$l_1 = -1 - (r_1/2), \quad l_2 = 1 + (r_2/2) \quad (27)$$

where r_1 and r_2 are the radii of curvature at these ends. Refinements of this rule, Eq. (27), are given by Landweber[9] and Moran.[10]

The stream function, obtained from expressions (7) and (8), assumes the alternative forms

$$\Psi(x, r) = \frac{r^2}{2} - \int_{-1}^1 \frac{m(\xi)(x-\xi)d\xi}{[(x-\xi)^2 + r^2]^{1/2}} + \frac{2r}{\pi}\int_0^\infty\int_{-1}^1 m(\xi)\frac{K_1(ka)}{I_1(ka)}I_1(kr)\sin k(x-\xi)d\xi dk \quad (28)$$

or

$$\Psi(x, r) = \frac{r^2}{2} - \frac{2r}{\pi}\int_0^\infty\int_{-1}^1 \left[K_1(kr) - \frac{K_1(ka)}{I_1(ka)}I_1(kr)\right]m(\xi)\sin k(x-\xi)d\xi dk \quad (29)$$

which yield Fredholm integral equations of the first kind when the boundary condition

$$\Psi[x, r(x)] = 0 \quad (30)$$

is introduced.

A. von Kármán method

Three different methods, proposed by von Kármán,[11] Kaplan,[12] and Landweber,[9] for treating the flow in an unbounded fluid ($a = \infty$), may be used for solving these integral equations. In von Kármán's method, the range of integration is divided into subintervals, in each of which $m(\xi)$ is treated as a constant, a mean value for that interval. The integrals in Eq. (28) can then be evaluated for each subinterval, the first in closed form, the second numerically, to obtain a set of linear equations for the unknown set of constant values of $m(\xi)$. Since a set of mean values of $m(\xi)$ is used, the accuracy of the integrations cannot be better than that of the trapezoidal rule. Furthermore, since in general an exact solution for $m(\xi)$ does not exist, one cannot improve the accuracy of this procedure by making the subintervals very small. Another disadvantage of the von Kármán method is that the resulting set of linear equations has a weak principal diagonal and, hence, is not suitable for solution by means of a rapidly converging interation formula.

B. Kaplan method

In Kaplan's method,[12] the source distribution is taken to be of the form of a series of Legendre polynomials

$$m(\xi) = \sum_{n=1}^\infty a_n P_n(\xi) \quad (31)$$

The first integral in Eq. (28) can then be expressed as a series of Legendre functions, the second integral as a series involving Bessel functions, with the a_n's as coefficients. Truncating these series then yields a set of linear equations for determining the a_n's, and hence $m(\xi)$.

Kaplan's expression for the stream function of the source distribution, the first integral in Eq. (28), with $m(\xi)$ in the form of Eq. (31), gives for the nth term of the series (assuming uniform convergence)

$$\left.\begin{aligned}\int_{-1}^1 \frac{P_n(\xi)(x-\xi)d\xi}{[(x-\xi)^2 + r^2]^{1/2}} &= \frac{2r^2}{n(n+1)}\dot{P}_n(\mu)\dot{Q}_n(\zeta), \quad n > 0 \\ &= -\frac{2n(n+1)}{(2n+1)^2}[P_{n+1}(\mu) - P_{n-1}(\mu)][Q_{n+1}(\zeta) - Q_{n-1}(\zeta)]\end{aligned}\right\} \quad (32)$$

in which the dot denotes differentiation with respect to the indicated argument and $Q_n(\zeta)$ is the Legendre function of the second kind. Here μ and ζ are confocal, prolate spheroidal coordinates, defined by

$$x = \mu\zeta, \quad r^2 = (1-\mu^2)(\zeta^2 - 1) \quad (33)$$

Relation (32) may be derived more expeditiously than was done by Kaplan, by first writing the nth symmetric spheroidal harmonic in the form given by the Havelock formula[13]

$$\frac{1}{2}\int_{-1}^1 \frac{P_n(\xi)d\xi}{[(x-\xi)^2 + r^2]^{1/2}} = P_n(\mu)Q_n(\zeta) \quad (34)$$

and then equating the corresponding stream functions, of which that for the right member is given by Lamb [see Ref. 14, p. 141].

The second integral in Eq. (28) can also be reduced by substituting the series (31) for $m(\xi)$ and applying the expression for the spherical Bessel functions $j_n(k)$ (see Ref. 7, p. 438),

$$j_n(k) = \frac{1}{2}(-i)^n \int_{-1}^{1} e^{ik\xi} P_n(\xi) d\xi \tag{35}$$

With this, the integral equation given by Eqs. (30) and (28) assumes the form

$$\sum_{n=1}^{\infty} C_n(x) a_n = \frac{1}{4} r(x) \tag{36}$$

where

$$C_n(x) = \frac{r(x)}{n(n+1)} \dot{P}_n[\mu(x)] \dot{Q}_n[\zeta(x)] - \frac{2}{\pi} \int_0^\infty \frac{K_1(ka)}{I_1(ka)} I_1[kr(x)] j_n(k) \cdot \mathrm{Re}(i^{n+1} e^{-ikx}) dk \tag{37}$$

and Re denotes "the real part of." In practice, the series (36) is truncated at, say, $n = N$ and the C_n's are evaluated for N values of $x = x_m$, defining a square matrix

$$C_{mn} = C_n(x_m)$$

With $r = r(x_m)$, Eq. (36) becomes the set of N equations

$$\sum_{n=1}^{N} C_{mn} a_n = (1/4) r_m \tag{38}$$

Once Eqs. (38) have been solved for the a_n's, both the potential and the stream function can be readily expressed in terms of them. For calculating the velocity field, it appears to be simpler to employ the former, which, by Eqs. (1) and (5), and the Havelock formula (34), becomes

$$\Phi(x, r) = x - 2 \sum_{n=1}^{N} a_n \left[P_n(\mu) Q_n(\zeta) + \frac{2}{\pi} \int_0^\infty \frac{K_1(ka)}{I_1(ka)} I_0(kr) j_n(k) \cdot \mathrm{Re}(i^n e^{-ikx}) dk \right] \tag{39}$$

Since

$$\left. \begin{aligned} \frac{\partial \mu}{\partial x} &= \frac{\zeta(1-\mu^2)}{\zeta^2 - \mu^2}, & \frac{\partial \zeta}{\partial x} &= \frac{\mu(\zeta^2 - 1)}{\zeta^2 - \mu^2} \\ \frac{\partial \mu}{\partial r} &= \frac{-\mu r}{\zeta^2 - \mu^2}, & \frac{\partial \zeta}{\partial r} &= \frac{\zeta r}{\zeta^2 - \mu^2} \end{aligned} \right\} \tag{40}$$

Eq. (39) yields, after some simplification

$$u = \frac{\partial \Phi}{\partial x} = 1 + 2 \sum_{n=1}^{N} a_n \left[\frac{n(n+1)}{2n+1} \times \frac{P_{n+1}(\mu) Q_{n-1}(\zeta) - P_{n-1}(\mu) Q_{n+1}(\zeta)}{\zeta^2 - \mu^2} + \frac{2}{\pi} \int_0^\infty \frac{K_1(ka)}{I_1(ka)} I_0(kr) k j_n(k) \cdot \mathrm{Re}(i^{n+1} e^{-ikx}) dk \right] \tag{41}$$

$$v = \frac{\partial \Phi}{\partial r} = \sum_{n=1}^{\infty} a_n \left\{ \frac{2r}{\zeta^2 - \mu^2} [\dot{P}_{n-1}(\mu) Q_n(\zeta) - P_n(\mu) \dot{Q}_{n-1}(\zeta)] - \frac{2}{\pi} \int_0^\infty \frac{K_1(ka)}{I_1(ka)} I_1(kr) k j_n(k) \cdot \mathrm{Re}(i^n e^{-ikx}) dk \right\} \tag{42}$$

Another set of linear equations for the a_n's can be derived from the form of the stream function (29). Substituting the series (31) for $m(\xi)$ and applying expression (35) for the spherical Bessel functions, we again obtain the set of linear equations (38), but with C_{mn} given by

$$C_{mn} = \frac{2}{\pi} \int_0^\infty \left[K_1(kr_m) - \frac{K_1(ka)}{I_1(ka)} I_1(kr_m) \right] j_n(k) \cdot \mathrm{Re}(i^{n+1} e^{ikx}) dk \tag{43}$$

The corresponding form of the potential, derived from Eq. (6), is

$$\Phi(x, r) = x - \frac{4}{\pi} \int_0^\infty \left[k_0(kr) + \frac{K_1(ka)}{I_1(ka)} I_0(kr) \right] \sum_{n=1}^{N} a_n j_n(k) \cdot \mathrm{Re}(i^n e^{-ikx}) dk \tag{44}$$

and the velocity components are now given by

$$u = 1 + \frac{4}{\pi} \int_0^\infty \left[K_0(kr) + \frac{k_1(ka)}{I_1(ka)} I_0(kr) \right] k \sum_{n=1}^{N} a_n j_n(k) \cdot \mathrm{Re}(i^{n+1} e^{-ikx}) dk \tag{45}$$

$$v = \frac{4}{\pi} \int_0^\infty \left[K_1(kr) - \frac{K_1(ka)}{I_1(ka)} I_1(kr) \right] k \sum_{n=1}^{N} a_n j_n(k) \cdot Re(i^n e^{-ikx}) dk \tag{46}$$

For numerical calculations, the previous forms in terms of the Legendre functions seem preferable.

C. Iteration procedure

A third method, in which an approximate numerical solution of an integral equation of the first kind is obtained by means of an iteration formula,[15] is better suited to the form of the integral equation when the disturbance flow is assumed to be generated by an axial doublet distribution. The reason for this is that the set of linear equations that results from approximating the integrals by quadrature formulas is much better conditioned for the doublet than for the source distribution. The integral equations are, however, equivalent since the stream function for the doublet distribution is derived from that for the source by differentiation with respect to x, and, denoting the doublet distribution by $M(x)$, the two distributions are related by

$$m(x) = dM/dx \tag{47}$$

when $M(1) = M(-1) = 0$.

Axial Doublet Distribution

The integral equation for a distribution of strength $-M(x)$ of x-oriented doublets on the x-axis in the range $-1 \le x \le 1$, obtained from Eq. (11), is

$$\int_{-1}^{1} \frac{M(\xi) r^2(x) d\xi}{[(x-\xi)^2 + r^2(x)]^{3/2}} - \frac{2r(x)}{\pi} \int_0^\infty \int_{-1}^{1} M(\xi) \frac{K_1(ka)}{I_1(ka)} I_1(kr) k \cos k(x-\xi) d\xi dk = \frac{1}{2} r^2(x) \tag{48}$$

The more elegant form yielded by Eq. (29) is less convenient for numerical evaluation, and hence it will not be presented.

A. von Kármán method

As for the source distribution, an approximate solution of Eq. (48) may be obtained by the von Kármán procedure of assuming that $M(\xi)$ is constant in each of a set of equal subintervals $x_{t-1} < \xi < x_t$, of length $\Delta x = 2/N$, in which $M(\xi) = M_t$ is constant, and $\xi_O = -1$, $\xi_N = 1$. The integral equation then reduces to the set of linear equations

$$\sum_{t=1}^{N} A_{st} M_t = \frac{1}{2} r_s^2 \tag{49}$$

where

$$A_{st} = \frac{x_t - x_s}{[(x_t - x_s)^2 + r_s^2]^{1/2}} - \frac{x_{t-1} - x_s}{[(x_{t-1} - x_s)^2 + r_s^2]^{1/2}} - \frac{4r_s}{\pi} \int_0^\infty \frac{K_1(ka)}{I_1(ka)} I_1(kr_s) \sin\frac{k}{N} \cos k(x_s - x_t + \frac{1}{N}) dk \quad (50)$$

For an elongated body, the matrix A_{st} has a strong principal diagonal, so that Eqs. (49) can be solved by an iterative procedure, in contrast with the linear equations for the source distribution.

The velocity potential, obtained from Eq. (9), is given by

$$\phi = Ux + \int_{-1}^{1} \frac{M(\xi)(x - \xi)d\xi}{[(x - \xi)^2 + r^2]^{3/2}} + \frac{2}{\pi} \int_0^\infty \int_{-1}^{1} \frac{M(\xi)K_1(ka)I_0(kr)}{I_1(ka)} k \sin k(x - \xi) dk d\xi \quad (51)$$

In terms of the values M_t derived from Eq. (49), this becomes

$$\phi = Ux + \sum_{t=1}^{N} M_t \left\{ \frac{1}{[(x_t - x)^2 + r^2]^{1/2}} - \frac{1}{[(x_{t-1} - x)^2 + r^2]^{1/2}} - \frac{4}{\pi} \int_0^\infty \frac{K_1(ka)I_0(kr)}{I_1(ka)} \sin\left[k\left(x - x_t + \frac{1}{N}\right)\right] \sin\frac{k}{N} dk \right\} \quad (52)$$

which yields the velocity components, $u = \partial\phi/\partial x$, $v = \partial\phi/\partial r$, as

$$u = U + \sum_{t=1}^{N} M_t \left\{ \frac{x_t - x}{[(x_t - x)^2 + r^2]^{3/2}} - \frac{x_{t-1} - x}{[(x_{t-1} - x)^2 + r^2]^{3/2}} - \frac{4}{\pi} \int_0^\infty \frac{K_1(ka)I_0(kr)}{I_1(ka)} k \cos\left[k\left(x - x_t + \frac{1}{N}\right)\right] \sin\frac{k}{N} dk \right\} \quad (53)$$

$$v = -\sum_{t=1}^{N} M_t \left\{ \frac{r}{[(x_t - x)^2 + r^2]^{3/2}} - \frac{r}{[(x_{t-1} - x)^2 + r^2]^{3/2}} + \frac{4}{\pi} \int_0^\infty \frac{K_1(ka)I_1(kr)}{I_1(ka)} k \sin\left[k\left(x - x_t + \frac{1}{N}\right)\right] \sin\frac{k}{N} dk \right\} \quad (54)$$

B. Iteration procedure

The preceding solution can be improved by fitting the integrals in Eq. (48) more closely with quadrature formulas. This is the essential advantage of the method employing an iteration formula[15] for solving the integral equation. In practice, a modified form of the iteration formula is used, derived for elongated bodies by treating the first part of the kernel of Eq. (48), $r^2(x)/[(x - \xi)^2 + r^2(x)]^{3/2}$, approximately as a Dirac delta function. As shown in Ref 4, the resulting iteration formula is

$$M_{n+1}(x) = M_n(x) + \frac{1}{4} r^2(x) - \frac{1}{2} \int_{-1}^{1} M_n(\xi) H(x, \xi) d\xi \quad (55)$$

where

$$H(x, \xi) = \frac{r^2(x)}{[(x - \xi)^2 + r^2(x)]^{3/2}} - \frac{2r(x)}{\pi} \int_0^\infty k \frac{K_1(ka)I_1[kr(x)]}{I_1(ka)} \cos k(x - \xi) dk \quad (56)$$

beginning with a first approximation[9]

$$M_1(x) = (1/4)(1 + k_1) r^2(x) \quad (57)$$

Here k_1, the longitudinal added-mass coefficient, can be estimated from the value for an "equivalent" spheroid, i.e., one of the same length and volume as the given body.

In order to eliminate the peak in the neighborhood of $x = \xi$ so that the integral can be fitted with a quadrature formula of moderate order, we write

$$\int_{-1}^{1} \frac{M_n(\xi) r^2(x) d\xi}{[(x - \xi)^2 + r^2(x)]^{3/2}} = \int_{-1}^{1} \frac{[M_n(\xi) - M_n(x)] r^2(x)}{[(x - \xi)^2 + r^2(x)]^{3/2}} d\xi + M_n(x) \left\{ \frac{1 + x}{[(1 + x)^2 + r^2(x)]^{1/2}} + \frac{1 - x}{[(1 - x)^2 + r^2(x)]^{1/2}} \right\} \quad (58)$$

Also, if $M_n(x)$, assumed to be determined by the iteration procedure, is fitted by a series of Legendre polynomials,

$$M_n(\xi) = \sum_{m=0}^{N} A_{mn} P_m(\xi) \quad (59)$$

the second integral of the integral equation can be reduced to a simple integral by a now familiar procedure. An algorithm for determining the A_{mn}'s is given in Ref. 4. The resulting discretized form of the iteration formula is

$$M_{n+1,i} = M_{ni} + \frac{1}{4} r_i^2 - \frac{1}{2} \sum_{j=1}^{N} \lambda_j \frac{(M_{nj} - M_{ni}) r_i^2}{[(x_i - x_j)^2 + r_i^2]^{3/2}} - \frac{M_{ni}}{2} \left\{ \frac{1 + x_i}{[(1 + x_i)^2 + r_i^2]^{1/2}} + \frac{1 - x_i}{[(1 - x_i)^2 + r_i^2]^{1/2}} \right\} + \frac{2}{\pi} r_i \int_0^\infty \frac{K_1(ka)}{I_1(ka)} k I_1(kr_i) \sum_{m=0}^{N} A_{mn} j_m(k) \cdot Re(i^n e^{-ikx_i}) dk \quad (60)$$

in which a quadrature formula of order N with weighting factors λ_j is employed.

The resulting set of values $M(x_i) = M_i$, and of A_m given by $M(x) = \Sigma A_m P_m(x)$ can then be used to determine the velocity components from the expression for the potential. This can be derived from Eq. (39) by differentiating the expression for the disturbance potential with respect to x, but noting that $A_0 \neq 0$, to obtain

$$\Phi(x, r) = x - 2\frac{\partial}{\partial x} \sum_{n=0}^{N} A_n \left[P_n(\mu) Q_n(\zeta) + \frac{2}{\pi} \int_0^\infty \frac{K_1(ka)}{I_1(ka)} I_0(kr) j_n(k) \times Re(i^n e^{-ikx}) dk \right]$$

Here the positive, rather than the negative, derivative is required since the strength of the doublet distribution is $-M(x)$. This derivative has already been evaluated in Eq. (41), except for $n = 0$. The potential may then be expressed in the form

$$\Phi(x, r) = x + 2A_0 \left[\frac{\mu}{\zeta^2 - \mu^2} + \frac{2}{\pi} \int_0^\infty \frac{K_1(ka)}{I_1(ka)} I_0(kr) \sin k \sin kx \, dk \right] + 2\sum_{n=1}^{N} A_n \times \left\{ \frac{n(n + 1)}{(2n + 1)(\zeta^2 - \mu^2)} [P_{n+1}(\mu) Q_{n-1}(\zeta) - P_{n-1}(\mu) Q_{n+1}(\zeta)] + \frac{2}{\pi} \int_0^\infty \frac{K_1(ka)}{I_1(ka)} I_0(kr) k j_n(k) \cdot Re(i^{n+1} e^{-ikx}) dk \right\} \quad (61)$$

Applying Eq. (40), this yields

$$u = 1 + 2A_0 \left[\frac{\zeta(\zeta^2 + 3\mu^2 - \mu^4 - 3\mu^2\zeta^2)}{(\zeta^2 - \mu^2)^3} + \frac{2}{\pi} \int_0^\infty \frac{K_1(ka)}{I_1(ka)} I_0(kr) k \sin k \cos kx \, dk \right] + 2\sum_{n=1}^{N} A_n \left\{ \frac{n(n + 1)}{(2n + 1)(\zeta^2 - \mu^2)^2} \left[\frac{2x(\zeta^2 + \mu^2 - 2)}{\zeta^2 - \mu^2} \times (P_{n-1} Q_{n+1} - P_{n+1} Q_{n-1}) + \zeta(1 - \mu^2)(\dot{P}_{n+1} Q_{n-1} - \dot{P}_{n-1} Q_{n+1}) + \mu(\zeta^2 - 1)(P_{n+1} \dot{Q}_{n-1} - P_{n-1} \dot{Q}_{n+1}) \right] + \frac{2}{\pi} \int_0^\infty \frac{K_1(ka)}{I_1(ka)} I_0(kr) k^2 j_n(k) \cdot Re(i^n e^{-ikx}) dk \right\} \quad (62)$$

$$v = 2A_0\left[-\frac{\mu r(3\zeta^2+\mu^2)}{(\zeta^2-\mu^2)^3} + \frac{2}{\pi}\int_0^\infty \frac{K_1(ka)}{I_1(ka)} I_1(kr)k \sin k \sin kx dk\right] + 2\sum_{n=1}^{N} A_n \times\left\{\frac{rn(n+1)}{(2n+1)(\zeta^2-\mu^2)^2}\left[2\frac{\zeta^2+\mu^2}{\zeta^2-\mu^2}(P_{n-1}Q_{n+1}-P_{n+1}Q_{n-1}) + \mu(\dot{P}_{n-1}Q_{n+1}-\dot{P}_{n+1}Q_{n-1}) - \zeta(P_{n-1}\dot{Q}_{n+1}-P_{n+1}\dot{Q}_{n-1})\right] + \frac{2}{\pi}\int_0^\infty \frac{K_1(ka)}{I_1(ka)} I_1(kr)k^2 j_n(k)\cdot Re(i^{n+1}e^{-ikx})dk\right\} \quad (63)$$

in which the P's are functions of μ and the Q's are functions of ζ.

C. Kaplan method

Here also Kaplan's procedure of transforming the integral equation (48) into a set of linear equations for the A_n's is applicable. Applying the Havelock formula,[13] we obtain

$$\int_{-1}^{1} \frac{P_n(\xi)r^2 d\xi}{[(x-\xi)^2+r^2]^{3/2}} = -r\frac{\partial}{\partial r}\int_{-1}^{1}\frac{P_n(\xi)d\xi}{[(x-\xi)^2+r^2]^{1/2}} = -2r\frac{\partial}{\partial r}P_n(\mu)Q_n(\zeta) = \frac{2r^2}{\zeta^2-\mu^2}(\mu\dot{P}_nQ_n - \zeta P_n\dot{Q}_n) \quad (64)$$

Hence, assuming that

$$M(\xi) = \sum_{n=0}^{\infty} A_n P_n(\xi) \quad (65)$$

is a uniformly convergent series, Eq. (48) yields the truncated set of linear equations

$$\sum_{n=0}^{N} B_{mn}A_n = (1/4)r_m \quad (66)$$

where, by Eq. (61)

$$B_{mn} = \frac{r_m}{\zeta_m^2-\mu_m^2}[\mu_m\dot{P}_n(\mu_m)Q_n(\zeta_m) - \zeta_m P_n(\mu_m)\dot{Q}_n(\zeta_m)] - \frac{2}{\pi}\int_0^\infty \frac{K_1(ka)}{I_1(ka)} I_1(kr_m)kj_n(k)\cdot Re(i^n e^{-ikx})dk \quad (67)$$

When the A_m's have been determined, the potential and the velocity components are given by Eqs. (61–63).

Vortex Sheet on Body Surface

The boundary condition that the body be a stream surface can be satisfied by assuming that a vortex sheet of strength $-u_s$ is distributed over the body surface, such that the fluid within the body is at rest. Here u_s is the velocity at the exterior side of the body surface, positive in the downstream sense. The vorticity at a point is perpendicular to the surface velocity, positive in the sense of a right-hand screw directed along the x-axis.

The desired integral equation is obtained from the condition that the resultant velocity at an interior point of the axis is zero. Applying the Biot-Savart law, we find for the contribution of the vortex sheet,

$$-\frac{1}{2}\int_{l_1}^{l_2}\frac{u_s(\xi)\sec\gamma(\xi)r^2(\xi)}{[(x-\xi)^2+r^2(\xi)]^{3/2}}d\xi$$

where $\gamma = \arctan dr/dx$. From the disturbance potential due to the tube wall, the second term of the integrand in Eq. (25), we obtain the contribution

$$-\frac{1}{\pi}\int_{l_1}^{l_2}\int_0^\infty u_s(\xi)\sec\gamma(\xi)r(\xi)\frac{K_1(ka)}{I_1(ka)}I_1[kr(\xi)]k\cos k(x-\xi)dkd\xi$$

Putting $g(\xi) = u_s(\xi)\sec\gamma(\xi)$, these yield the integral equation

$$\int_{l_1}^{l_2}\frac{g(\xi)r^2(\xi)d\xi}{[(x-\xi)^2+r^2(\xi)]^{3/2}} - \frac{2}{\pi}\int_{l_1}^{l_2}\int_0^\infty g(\xi)r(\xi)\frac{K_1(ka)}{I_1(ka)}I_1[kr(\xi)]k\cos k(x-\xi)dkd\xi = 2 \quad (68)$$

An important advantage of this integral equation over the previous ones is that it certainly has a solution, since the velocity distribution exists. One sees that it cannot be successfully treated by either the von Kármán or the Kaplan procedures. It is, however, well suited for application of an iteration formula, especially for an elongated body. For the special case of the prolate spheroid, however, Kaplan's method can be used, as has been shown by Miloh,[16] who collaborated with the present writer on this problem.

For such a body, the first integrand of Eq. (64) has a sharp peak in the neighborhood of $x = \xi$. A procedure for removing this peak, so that the resulting integral may be accurately represented by a quadrature formula of moderate order, and an interation formula for the numerical solution of the resulting equation are given in Ref. 4 for a modified form of this integral equation. This yields the discretized interation formula

$$g_{n+1,i} = g_{ni} + 1 - \sum_{j=1}^{N}\lambda_j[g_{nj}(K_{ji}-K_{ji}') - g_{ni}K_{ij}] - \frac{1}{2}g_{ni}\left\{\frac{1+x_i}{[(1+x_i)^2+r_i^2]^{1/2}} + \frac{1-x_i}{[(1-x_i)^2+r_i^2]^{1/2}}\right\} - \frac{g_{n1}}{2l_1-r_1+2x_i}\left\{l_1+x_i - \frac{(1+r_1+x_i)(l_1-r_1+x_i)+r_1^2}{[(1+x_i)^2+2r_1(l_1-1)]^{1/2}}\right\} - \frac{g_{nT}}{2l_2-r_2-2x_i}\left\{l_2-x_i-\frac{(1+r_2-x_i)(l_2-r_2-x_i)+r_2^2}{[(1-x_i)^2+2r_2(l_2-1)]^{1/2}}\right\} \quad (69)$$

where

$$K_{ij} = \frac{(1/2)r_i^2}{[(x_i-x_j)^2+r_i^2]^{3/2}}$$

and

$$K_{ij}' = \frac{1}{\pi}r_j\int_0^\infty\frac{K_1(ka)}{I_1(ka)}I_1(kr_j)k\cos k(x_i-x_j)dk$$

which differs from that given in Ref. 4, where the term $\Sigma\lambda_j K_{ji}'$ was determined from a separate solution for the axial doublet distribution. A suitable first approximation g_{1i} in Eq. (65) is

$$g_{1i} = 1 + k_1$$

as is shown in Ref. 9. Here k_1 is the added-mass coefficient defined in Eq. (57).

Discussion

Three methods have been considered for solving each of three Fredholm integral equations of the first kind for determining the irrotational axisymmetric flow about a body of revolution in a circular tube. For axial source or doublet distributions, any of the three methods may be used, but for the vortex sheet, only the iteration procedure is suitable for arbitrary bodies of revolution. For evaluating the efficiency and accuracy of these seven possible solutions, three were selected for numerical comparison, employing the same spheroid, of length-diameter ratio 6.33, as was used in Refs. 4 and 16.

Table 1 Coefficients[a] a_n

n	b/a = 0.50	0.80
1	−0.016515	−0.028870
3	0.003286	0.024107
5	−0.000912	−0.013405
7	−0.000008	0.005154
9	0.000014	−0.002443
11	0.000019	0.000847
13	0.000001	0.000179
15	−0.000008	−0.000284

[a] *Note:* $a_{2n} = 0$.

The three cases for which computer programs were written and executed are as follows. a) Solution for an axial source distribution by the Kaplan method. b) Solution for an axial doublet distribution by the von Kármán method. c) Solution for a vortex sheet by the iteration method.

The axial distributions were assumed to extend between the foci of the ellipse at $x = \pm 1$. For the first case, the source distribution was assumed in the form

$$m = \sum_{n=1}^{16} a_n P_n(\xi)$$

and the coefficients a_n were determined by solving the set of linear equations (38), in which the coefficients C_{mn}, defined in Eq. (37), were evaluated at the abscissas of the Gauss 16-point quadrature formula. The infinite integral in Eq. (37) was calculated by means of the Laguerre 15-point quadrature formula. The resulting values of a_n, for blockage ratios $b/a = 0.50$ and 0.80, are given in Table 1. These values were then applied to compute the velocity distribution on the spheroid ($\zeta = 1.0127$) by employing the expressions for the velocity components in Eqs. (41) and (42). The results are plotted in Fig. 1.

For determining the axial doublet distribution by the von Kármán method, the interval $-1 \leq \xi \leq 1$ was subdivided into $N = 16$ equal subintervals and the piecewise constant doublet distribution was computed from the set of linear equations (49) and (50). The coefficients A_{st} of these equations were evaluated at the 16 midpoints of these subintervals. The resulting values of M_t are given in Table 2. Equations (53) and (54) were then applied to compute the velocity distribution on the spheroid from $u_s^2 = u^2 + v^2$ (see Fig. 1). The results are seen to be in good agreement with the previous case except at the first or last points. In the neighborhood of the extremities, the computed value of the radial velocity component v is sensitive to the local variation of the doublet distribution and a large error is incurred when a mean value of M is used in the first or last interval. The longitudinal velocity component is much less sensitive to the local variation of M, and when the velocity on the body u_s is computed from the relation

$$u_s = u \sec\gamma, \quad \gamma = \arctan dr/dx \tag{70}$$

the results obtained are in much better agreement with those for the other cases.

Table 2 Doublet strength[a] $M(x)$

x-interval	b/a = 0.50	b/a = 0.80
0 to 0.125	0.0008047	0.0008534
0.125 0.250	0.0025133	0.0028711
0.250 0.375	0.0040337	0.0051804
0.375 0.500	0.0054377	0.0079124
0.500 0.625	0.0066598	0.0109513
0.625 0.750	0.0076451	0.0140928
0.750 0.875	0.0083375	0.0168060
0.875 1.000	0.0086952	0.0184301

[a] *Note:* $M(x) = M(-x)$

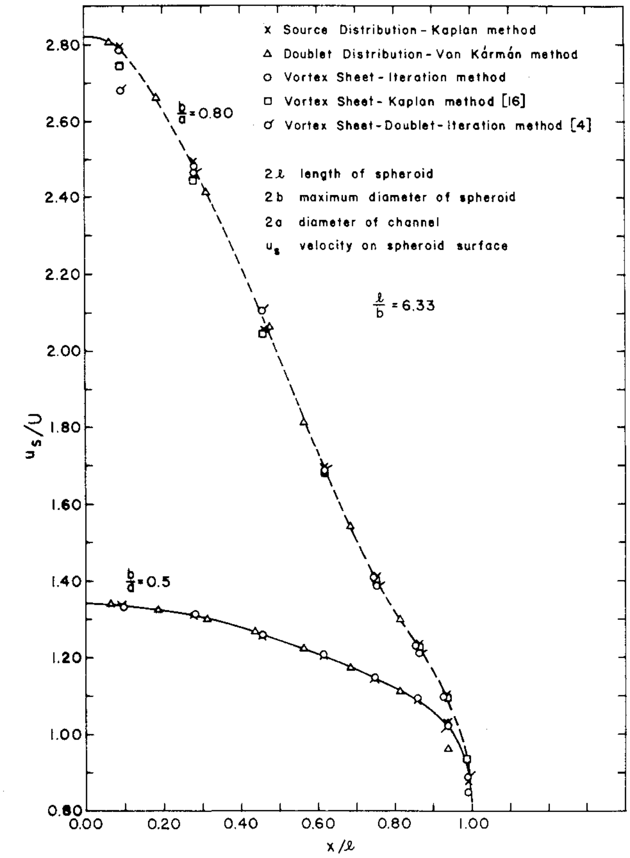

Fig. 1 Blockage effect on velocity distribution on a spheroid.

Lastly, the integral equation (68) for a vortex sheet on the body surface was solved by means of the iteration formula (69). The Gauss 16-point quadrature formula was used to transform the integral equation into a set of linear equations. In this case, the solution of the equations yields the velocity distribution on the body almost directly. It is seen from Fig. 1 that the results by this method agree well with the others. The values computed at the abscissas of the Gauss 16-point quadrature formula are given in Table 3.

Also shown in Fig. 1 are Miloh's[16] from the integral equation for the vortex sheet (68), solved by the Kaplan method by a procedure suitable only for the spheroid, and the earlier results of Landweber and Gopalakrishnan,[4] employing simultaneously the integral equations for the axial doublet distribution and vortex sheet. These are compared with the present results for the high-blockage condition $b/a = 0.80$. Miloh's values are seen to be slightly lower near the middle of the body. The results of Ref. [4] agree well with the other values except for the one point nearest the center of the body.

Of the three present methods applied to the spheroid,

Table 3 Velocity distribution from vortex sheet[a]

		b/a = 0.50		b/a = 0.80	
x	$r(x)$	$g(x)$	u_s/U	$g(x)$	u_s/U
0.095013	0.159281	1.34138	0.84796	2.80086	0.87697
0.281604	0.153677	1.31366	1.02775	2.49433	1.09844
0.458017	0.142690	1.26632	1.09120	2.07434	1.23699
0.617876	0.126759	1.21204	1.14564	1.70082	1.40903
0.655404	0.106577	1.16344	1.20317	1.43091	1.68837
0.865631	0.083037	1.12752	1.26227	1.27817	2.06802
0.944575	0.057692	1.11023	1.31229	1.18660	2.49172
0.989401	0.034133	1.04659	1.34123	1.08239	2.80055

[a] *Note:* $(u_s/U)(x) = (u_s/U)(-x)$: $r(x) = [1/(\lambda^2 - 1) - x^2/\lambda^2]^{1/2} = [0.0255958 - 0.0249570x^2]^{1/2}$; and $\lambda = 6.33$.

Table 4 Added-mass coefficients k_1

	$b/a = 0.50$	$b/a = 0.80$
Source distribution, Kaplan method	0.2742	1.2276
Doublet distribution, von Kármán method	0.2768	1.2307
Vortex sheet, iteration method	0.2754	1.2336

that of von Kármán for the axial doublet distribution was about as fast as the solution for the vortex sheet by iteration. Computer time for solving the axial source distribution by the Kaplan method was about 40% greater than that by the von Kármán method. The latter yielded poor accuracy for the velocity distribution near the ends of the body, although, as discussed previously, acceptable accuracy can be obtained by calculating u_s from Eq. (70), instead of as the resultant of both velocity components.

The favorable results from axial source and doublet distributions is partly attributable to the fact that exact solutions for these distributions probably exist for the spheroid. For other bodies of revolution, exact solutions for the axial distributions do not exist in general, and one could only expect to find approximate solutions which nearly satisfy the integral equations in a least-square sense.[15] In contrast, the integral equation for the vortex sheet does possess an exact solution, although there is no assurance that the indicated iteration procedure can yield more than convergence-in-the-mean to this solution.[15]

The foregoing considerations offer little basis for preferring one method over another, other than a slight trade-off of accuracy vs economy. A comparison of the added-mass coefficients, computed by the various methods as described in the Appendix, also shows only a slight variation of less than 1%. For a body of revolution which is not generated by an axial distribution, however, the agreement of the added masses would probably be less favorable, as is indicated by Munk's discussion of Kaplan's paper.[12] In the main, then, what has been demonstrated is that there are many methods, employing Fredholm integral equations of the first kind, which can be used to calculate, with sufficient accuracy for most purposes, the axisymmetric irrotational flow about a body of revolution in a tube.

Appendix I: Added-Mass Coefficient

From the results for the axial source and doublet distributions and the vortex sheet, one can readily determine the added-mass coefficient of the body of revolution, $k_1 = A/\rho\mathcal{V}$, where A is the added mass, ρ the mass density of the fluid, and $\mathcal{V}$ the volume of the body. From the Taylor added-mass formula [see Ref. 14, p. 166], we have in the present case,

$$1 + k_1 = -\frac{4\pi}{\mathcal{V}}\int_{-1}^{1} xm(x)dx = \frac{4\pi}{\mathcal{V}}\int_{-1}^{1} M(x)dx \quad (71)$$

where $m(x)$ and $M(x)$ denote the axial source and doublet distributions, respectively. In terms of the velocity distribution we also have[9]

$$1 + k_1 = \frac{\pi}{\mathcal{V}}\int_{-l}^{l} u_s r^2(x)\sec\gamma\, dx = \frac{\pi}{\mathcal{V}}\int_{-l}^{l} g(x)r^2(x)dx \quad (72)$$

where $g(x) = u_s \sec\gamma$, as in Eq. (68).

Table 5 Comparison of results for $m(1)$ and $u(o)$

	Slender-body theory		Integral equation solution	
α	$m(1)$	$u(o)$	$m(1)$	$u(o)$
0.50	−0.0126	0.33	−0.0141	0.34
0.80	−0.0129	1.76	−0.0147	2.81

In terms of the length-diameter ratio of the body λ, we have for the spheroid,

$$\mathcal{V} = \frac{4\pi\lambda}{3(\lambda^2 - 1)^{3/2}}$$

When $\lambda = 6.33$, Eqs. (71) and (72) then become

$$1 + k_1 = -115.735\int_{-1}^{1} xm(x)dx = 115.735\int_{-1}^{1} M(x)dx \quad (71a)$$

and

$$k_1 = 28.93375\int_{-l}^{l}[g(x) - 1]r^2(x)dx \quad (72a)$$

Since the Kaplan method expresses $m(x)$ as a series of Legendre polynomials, $m(x) = \Sigma_{n=1}^{16} a_n P_n(x)$, we immediately obtain, from the orthogonality of P_n's,

$$1 + k_1 = -(8\pi/3\mathcal{V})a_1 \quad (73)$$

For the spheroid, this becomes

$$1 + k_1 = -\frac{2(\lambda^2 - 1)^{3/2}}{\lambda}a_1 = -77.1567\, a_1$$

when $\lambda = 6.33$. Substituting the values of a_1 from Table 1 then yields the values of k_1 given in Table 4.

According to the von Kármán method, we have

$$\int_{-1}^{1} M(x)dx = \frac{2}{N}\sum_{i=1}^{N} M_i, \quad N = 16$$

Then, summing the values of M_i given in Table 2 and substituting the resulting value of the integral in Eq. (71a), yields the values of K_1 for this method, given in Table 4.

Finally, since the values of $g(x)$ for the vortex sheet have already been computed at the abscissas of the Gauss 16-point quadrature formula, we obtain immediately from Eq. (72a)

$$k_1 = 28.93375\left[\sum_1^{16}\lambda_i(g_i - 1)r_i^2 + 2\int_1^{l}[g(x) - 1]r^2(x)dx\right.$$

$$\doteq 28.93375\left[\sum_1^{16}\lambda_i(g_i - 1)r_i^2 + \frac{g_1 - 1}{2\lambda^6}\right] \quad \lambda = 6.33$$

Here the last term is found to be negligible. Employing the values of g_i from Table 3, we obtain the values of k_1 given in Table 4.

Appendix II: Comparison with Slender-Body Theory

Expressions for the axial source distribution

$$m(x) = \frac{1}{2}\frac{r(dr/dx)}{(1 - (r^2/a^2))^2} \quad (74)$$

and for the axial velocity component at the body,

$$u = (2/a^2)\left[\int_{-1}^{x} m(\xi)d\xi - \int_{x}^{1} m(\xi)d\xi\right] \quad (75)$$

derived from slender-body theory, have been given by Goodman.[5] For the spheroid of length-diameter ratio λ,

$$r^2(x) = \frac{1}{\lambda^2 - 1} - \frac{x^2}{\lambda^2} \quad (76)$$

Eq. (74) becomes

$$m(x) = -\frac{x}{2\lambda^2}\left[1 - \alpha^2 + \frac{\alpha^2(\lambda^2 - 1)}{\lambda^2}x^2\right]^{-2} \quad (77)$$

where α is the ratio of the maximum diameter of the body to that of the tube; for the spheroid, we have

$$1/\alpha^2 = a^2(\lambda^2 - 1) \quad (78)$$

From Eqs. (75) and (77), we then obtain for the spheroid at $x = 0$,

$$u(o) = \frac{\alpha^2(\lambda^2 - 1)}{(1 - \alpha^2)(\lambda^2 - \alpha^2)} \tag{79}$$

Results given by Eq. (77) for $m(1)$ and for $u(o)$ from Eq. (70) are compared in Table 5, for $\lambda = 6.33$, with the solution of the integral equation given in Table 1 and in Fig. 1. The sum of the coefficients, Σa_n, in Table 1 gives $m(1)$ directly. It is seen that $m(1)$ given by slender-body theory is small by about 11% $u(o)$ is low by only 3% for $\alpha = 0.50$, but by 37% for $\alpha = 0.80$. This indicates that the slender-body formula for the velocity at the body would be useful for moderate blockage.

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